Quantum Polynomial Hierarchies: Karp-Lipton, Error Reduction, and Lower Bounds

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2 Results

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4 Error Reduction

5 Lower Bounds

6 Conclusion

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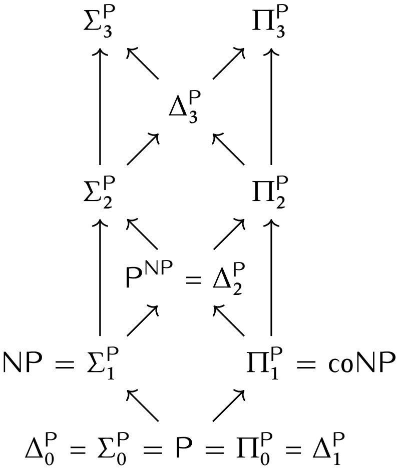
- Introduced by Stockmeyer 1976, staple of classical complexity theory.



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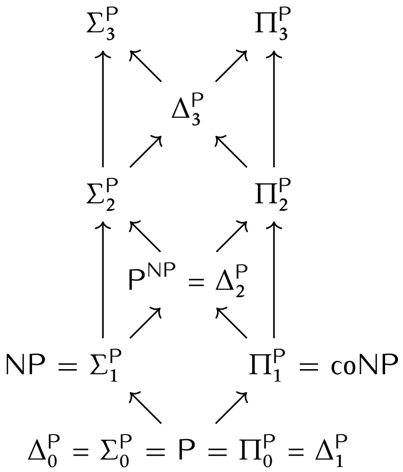

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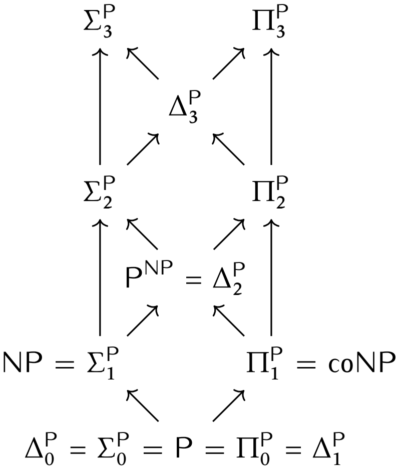
Definition (Σ_i^{P})

$L \in \Sigma_i^{\mathrm{P}}$ if there exists a deterministic poly-time Turing machine M s.t.

- $x \in L \Rightarrow \exists y_1 \forall y_2 \exists y_3 \cdots Q_i y_i : M(x, y_1, \ldots, y_i) = 1$
- $x \notin L \Rightarrow \forall y_1 \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i : M(x, y_1, \ldots, y_i) = 0$



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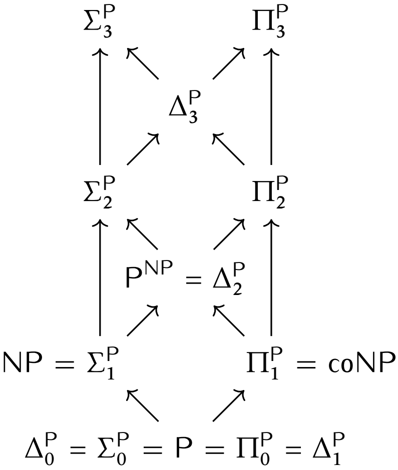
- Π_i^{P}: Same as Σ_i^{P}, but with inverted quantifiers



The *Quantum* Polynomial Hierarchy (QPH)

- Many ways to define a quantum polynomial hierarchy...

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Quantum Classical PH (QCPH) [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022]

Promise problem $L = (L_{\text{yes}}, L_{\text{no}}) \in \mathrm{QC}\Sigma_i$ if there exists a *poly-time uniform* family of quantum circuits $\{V_n\}_{n\in\mathbb{N}}$ and c, s with $c - s \geq 1/\operatorname{poly}(n)$ s.t.

- *Completeness:* $x \in L_{\text{yes}} \Rightarrow \exists y_1 \forall y_2 \cdots Q_i y_i : \Pr[V_n \text{ accepts } (x, y_1, \ldots, y_i)] \geq c$
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- QPH [GSSSY22]: proofs are mixed states
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- Error reduction possible for QCPH but not known for QPH, pureQPH.

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Promise problem $L = (L_{\mathrm{yes}}, L_{\mathrm{no}}) \in \mathsf{BQP}_{/\mathrm{mpoly}}$ if there exists a *poly-time uniform* family of quantum circuits $\{C_n\}_{n\in\mathbb{N}}$ and a collection of advice strings $\{a_n\}_{n\in\mathbb{N}}$ with $|a_n| = \mathrm{poly}(n)$ s.t.

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- Why? Same problem as ∃BPP vs. MA: $\mathsf{BQP}_{/\,\mathrm{poly}}$ has to accept with probability $\leq 1/3$ or $\geq 2/3$ **even for bad advice!**

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Collapse theorem for QCPH

If for any $k \geq 1$, $\mathrm{QC}\Pi_k = \mathrm{QC}\Sigma_k$, *then* $\mathrm{QCPH} = \mathrm{QC}\Sigma_k$.

- Also in concurrent work [Falor, Ge, Natarajan, 2023]

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- QCMA cannot be solved by (non-uniform) poly-size quantum circuits, unless QCPH collapses to its second level.

Results: Error Reduction

One-sided error reduction for pureQPH

For all $i > 0$ and $c - s \geq 1/p(n)$ for some polynomial p,

1. For even $i > 0$:

$$\text{pureQ}\Pi_i(c, s) \subseteq \text{pureQ}\Pi_i^{\text{SEP}}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right)$$

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- $\text{pureQ}\Sigma_i^{\text{SEP}} \subseteq \text{pureQ}\Sigma_i$ same as $\text{pureQ}\Sigma_i$, but YES-case POVM must be separable across all i proofs: $H = \sum_j H_1^{(j)} \otimes \cdots \otimes H_i^{(j)}$

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- **Significance:** One-sided error reduction for QMA(2) [Aaronson, Beigi, Drucker, Fefferman, Shor, 2008] is the first step in the two-sided error reduction [Harrow, Montanaro, 2012], also relying on verifier separability.

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  - Note, $\text{QMA}(2) \subseteq \text{Q}\Sigma_3 \subseteq \text{pureQ}\Sigma_3$.

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$\mathrm{QCPH} \subseteq \mathrm{pureQPH} \subseteq \mathrm{EXP}^{\mathrm{PP}}$

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Results: Upper & Lower Bounds

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Hence, $L \in \mathrm{QC}\Sigma_{i-1}$. □

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If $\mathrm{QCMA} \subseteq \mathrm{BQP}_{/\mathrm{mpoly}}$, *then* $\mathrm{QCPH} = \mathrm{QC}\Sigma_2 = \mathrm{QC}\Pi_2$.

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- We construct new "asymmetric version" of the product test [Mintert, Kuś, Buchleitner, 2005] [Harrow, Montanaro, 2012]

APT

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   - Register A: $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$ with $|\psi_i\rangle \in \mathbb{C}^{d_i}$
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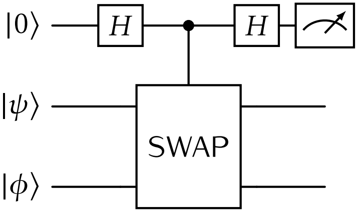

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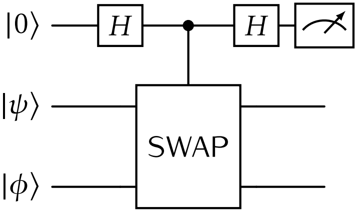

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Let $|\phi\rangle_{BC} \in \mathbb{C}^{d^m} \otimes \mathbb{C}^{d'}$ *for some* $d' > 0$ *s.t.*

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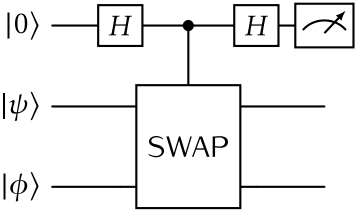

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for $\epsilon \geq 0$. Then, APT accepts $|\eta\rangle_{ABC} := |\psi\rangle_A \otimes |\phi\rangle_{BC}$ w.p. $\leq 1 - \epsilon/2mn$ for all $|\psi\rangle$.

One-sided Error Reduction

Lemma

Let i be even, $c - s \geq 1/p(n)$. Then

$$\mathrm{pureQ\Pi}_i(c, s) \subseteq \mathrm{pureQ\Pi}_i^{\mathsf{SEP}}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right).$$

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- Verifier $V'((|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_A \otimes |\psi_i'\rangle_{BC})$ with $|\psi_i'\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_B^{\otimes m} \otimes |\psi_i\rangle_C^{\otimes m}$ (for honest last prover). Do with probability $1/2$:

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- Note that APT also bounds entanglement between B, C.

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