# Quantum Polynomial Hierarchies: Karp-Lipton, Error Reduction, and Lower Bounds

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The Polun	omial Hie	erarchu (PH)			

• Introduced by Stockmeyer 1976, staple of classical complexity theory.



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## The Polynomial Hierarchy (PH)

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- Applications from *randomized computation* to *quantum advantage* analysis for near-term quantum computers.



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### Definition $(\Sigma_i^{\mathsf{P}})$

 $L \in \Sigma_i^{\mathsf{P}}$  if there exists a deterministic poly-time Turing machine M s.t.

• 
$$x \in L \Rightarrow \exists y_1 \forall y_2 \exists y_3 \cdots Q_i y_i \colon M(x, y_1, \dots, y_i) = 1$$
  
•  $x \notin L \Rightarrow \forall y_1 \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon M(x, y_1, \dots, y_i) = 0$ 



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•  $\Pi^{\mathsf{P}}_i$ : Same as  $\Sigma^{\mathsf{P}}_i$ , but with inverted quantifiers





## The *Quantum* Polynomial Hierarchy (QPH)

• Many ways to define a quantum polynomial hierarchy...

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The Quantu	m Polynon	nial Hierarchy (Q	PH)		

Quantum Classical PH (QCPH) [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022]

- Completeness:  $x \in L_{yes} \Rightarrow \exists y_1 \forall y_2 \cdots Q_i y_i \colon \Pr[V_n \text{ accepts } (x, y_1, \dots, y_i)] \ge c$
- Soundness:  $x \in L_{no} \Rightarrow \forall y_1 \exists y_2 \cdots \overline{Q_i} y_i$ :  $\Pr[V_n \text{ accepts } (x, y_1, \dots, y_i)] \le s$



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- QPH [GSSSY22]: proofs are mixed states  $x \in L_{yes} \Rightarrow \exists \rho_1 \forall \rho_2 \cdots Q_i \rho_i \colon \Pr[V_n \text{ accepts } |x\rangle\langle x| \otimes \rho_1 \otimes \cdots \otimes \rho_i] \ge c.$

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## The *Quantum* Polynomial Hierarchy (OPH)

Quantum Classical PH (QCPH) [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022]

Promise problem  $L = (L_{ves}, L_{no}) \in QC\Sigma_i$  if there exists a *poly-time uniform* family of quantum circuits  $\{V_n\}_{n \in \mathbb{N}}$  and c, s with  $c - s \ge 1/\operatorname{poly}(n)$  s.t.

- Completeness:  $x \in L_{ves} \Rightarrow \exists y_1 \forall y_2 \cdots Q_i y_i$ :  $\Pr[V_n \text{ accepts } (x, y_1, \dots, y_i)] \ge c$
- $x \in L_{n_0} \Rightarrow \forall v_1 \exists v_2 \cdots \overline{Q_i} v_i \colon \Pr[V_n \text{ accepts } (x, v_1, \dots, v_i)] \leq s$ Soundness:

• QPH [GSSSY22]: proofs are mixed states  $x \in L_{\text{vos}} \Rightarrow \exists \rho_1 \forall \rho_2 \cdots Q_i \rho_i \colon \Pr[V_n \text{ accepts } |x\rangle \langle x| \otimes \rho_1 \otimes \cdots \otimes \rho_i] \ge c.$ • pureOPH (this work): proofs are pure states  $x \in L_{\text{ves}} \Rightarrow \exists |\psi_1\rangle \forall |\psi_2\rangle \cdots Q_i |\psi_i\rangle \colon \Pr[V_n \text{ accepts } |x\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_i\rangle] \ge c.$  Introduction<br/>OOOResults<br/>OOOQuantum Karp-Lipton<br/>OOOError Reduction<br/>OOOLower Bounds<br/>OOConclusion<br/>OOThe Ouantum Polynomial Hierarchy (OPH)

### Quantum Classical PH (QCPH) [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022]

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## The *Quantum* Polynomial Hierarchy (QPH)

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  - Sion's minimax theorem:  $\max_{\rho} \min_{\sigma} \operatorname{Tr}(H(\rho \otimes \sigma)) = \min_{\sigma} \max_{\rho} \operatorname{Tr}(H(\rho \otimes \sigma))$

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Quantum Classical PH (QCPH) [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022]

- Completeness:  $x \in L_{yes} \Rightarrow \exists y_1 \forall y_2 \cdots Q_i y_i$ :  $\Pr[V_n \text{ accepts } (x, y_1, \dots, y_i)] \ge 1 \frac{1}{\exp}$
- Soundness:  $x \in L_{no} \Rightarrow \forall y_1 \exists y_2 \cdots \overline{Q_i} y_i$ :  $\Pr[V_n \text{ accepts } (x, y_1, \dots, y_i)] \leq \frac{1}{\exp}$
- QPH [GSSSY22]: proofs are mixed states

  x ∈ L<sub>yes</sub> ⇒ ∃ρ<sub>1</sub> ∀ρ<sub>2</sub> ··· Q<sub>i</sub>ρ<sub>i</sub>: Pr[V<sub>n</sub> accepts |x⟩⟨x| ⊗ ρ<sub>1</sub> ⊗ ··· ⊗ ρ<sub>i</sub>] ≥ c.

  pureQPH (this work): proofs are pure states

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  Sion's minimax theorem: max<sub>ρ</sub> min<sub>σ</sub> Tr(H(ρ ⊗ σ)) = min<sub>σ</sub> max<sub>ρ</sub> Tr(H(ρ ⊗ σ))
- Error reduction possible for QCPH but not known for QPH, pureQPH.

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• Karp-Lipton theorem: SAT cannot be solved with poly-size circuits, unless PH collapses to its second level.

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# Results: Quantum Karp-Lipton

- Karp-Lipton theorem: SAT cannot be solved with poly-size circuits, unless PH collapses to its second level.
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# Results: Quantum Karp<u>–Lipton</u>

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- GSSSY22: If Precise-QCMA  $\subseteq$  BQP $_{/mpoly}$ , then QC $\Pi_2 =$  QC $\Sigma_2$ .

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### $BQP_{/mpoly}$ (BQP with deterministic poly-size Merlin-like advice)

Promise problem  $L = (L_{\text{yes}}, L_{\text{no}}) \in \text{BQP}_{/\text{mpoly}}$  if there exists a *poly-time uniform* family of quantum circuits  $\{C_n\}_{n \in \mathbb{N}}$  and a collection of advice strings  $\{a_n\}_{n \in \mathbb{N}}$  with  $|a_n| = \text{poly}(n)$  s.t.

• 
$$x \in L_{\text{yes}} \Rightarrow \Pr[C_n \text{ accepts } |x\rangle|a_n\rangle] \ge 2/3$$

• 
$$x \in L_{no} \Rightarrow \Pr[C_n \text{ accepts } |x\rangle|a_n\rangle] \le 1/3$$

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- $x \in L_{\text{yes}} \Rightarrow \Pr[C_n \text{ accepts } |x\rangle|a_n\rangle] \ge 2/3$
- $x \in L_{no} \Rightarrow \Pr[C_n \text{ accepts } |x\rangle|a_n\rangle] \le 1/3$
- Why? Same problem as <code>∃BPP vs. MA: BQP\_/ poly</code> has to accept with probability  $\leq 1/3$  or  $\geq 2/3$  even for bad advice!

A. Agarwal, S. Gharibian, V. Koppula, D. Rudolph

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# Results: Quantum Karp-Lipton

- Karp-Lipton theorem: SAT cannot be solved with poly-size circuits, unless PH collapses to its second level.
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### Collapse theorem for QCPH

If for any  $k \ge 1$ ,  $QC\Pi_k = QC\Sigma_k$ , then  $QCPH = QC\Sigma_k$ .

• Also in concurrent work [Falor, Ge, Natarajan, 2023]

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### Karp-Lipton theorem for QCPH

If QCMA  $\subseteq$  BQP<sub>/mpoly'</sub> then QCPH = QC $\Sigma_2$  = QC $\Pi_2$ .

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#### Collapse theorem for QCPH

If for any  $k \ge 1$ ,  $QC\Pi_k = QC\Sigma_k$ , then  $QCPH = QC\Sigma_k$ .

### Karp-Lipton theorem for QCPH

If QCMA 
$$\subseteq$$
 BQP<sub>/mpoly'</sub> then QCPH = QC $\Sigma_2$  = QC $\Pi_2$ .

• QCMA cannot be solved by (non-uniform) poly-size quantum circuits, unless QCPH collapses to its second level.

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Results: I	- rror Redi	uction			

For all i > 0 and  $c - s \ge 1/p(n)$  for some polynomial p,

• For even i > 0:

pureQ
$$\Pi_i(c,s) \subseteq \text{pureQ}\Pi_i^{\text{SEP}}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right)$$

Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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Results:	Error Red	uction			

For all i > 0 and  $c - s \ge 1/p(n)$  for some polynomial p,

- For even i > 0:
  - pure  $Q\Sigma_i(c, s) \subseteq pure Q\Sigma_i^{SEP}(1/np(n)^2, 1/e^n)$
  - pure  $Q\Pi_i(c, s) \subseteq pure Q\Pi_i^{SEP}(1 1/e^n, 1 1/np(n)^2)$

### Solution For odd i > 0:

- pure  $Q\Sigma_i(c, s) \subseteq pure Q\Sigma_i^{SEP}(1 1/e^n, 1 1/np(n)^2)$
- ② pureQ $\Pi_i(c, s)$  ⊆ pureQ $\Pi_i^{\text{SEP}}(1/np(n)^2, 1/e^n)$

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Results:	Error Red	uction			

For all i > 0 and  $c - s \ge 1/p(n)$  for some polynomial p,

- For even i > 0:
  - pure  $Q\Sigma_i(c,s) \subseteq pure Q\Sigma_{i_{OUT}}^{SEP}(1/np(n)^2, 1/e^n)$
  - pureQ $\Pi_i(c,s) \subseteq \text{pureQ}\Pi_i^{\text{SEP}}(1-1/e^n, 1-1/np(n)^2)$
- ② For odd i > 0:
  - pure  $Q\Sigma_i(c,s) \subseteq pure Q\Sigma_i^{SEP}(1-1/e^n, 1-1/np(n)^2)$
  - ② pureQ $\Pi_i(c,s)$  ⊆ pureQ $\Pi_i^{\mathsf{SEP}}(1/np(n)^2, 1/e^n)$

• pureQ $\Sigma_i^{\text{SEP}} \subseteq \text{pureQ}\Sigma_i$  same as pureQ $\Sigma_i$ , but YES-case POVM must be separable across all *i* proofs:  $H = \sum_j H_1^{(j)} \otimes \cdots \otimes H_i^{(j)}$ 

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Results:	Error Red	uction			

For all i > 0 and  $c - s \ge 1/p(n)$  for some polynomial p,

- For even i > 0:
  - pure  $Q\Sigma_i(c, s) \subseteq pure Q\Sigma_i^{SEP}(1/np(n)^2, 1/e^n)$
  - pure  $Q\Pi_i(c,s) \subseteq pure Q\Pi_i^{SEP}(1-1/e^n, 1-1/np(n)^2)$
- ② For odd i > 0:
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  - ② pureQ $\Pi_i(c,s)$  ⊆ pureQ $\Pi_i^{\text{SEP}}(1/np(n)^2, 1/e^n)$
- **Significance:** One-sided error reduction for QMA(2) [Aaronson, Beigi, Drucker, Fefferman, Shor, 2008] is the first step in the two-sided error reduction [Harrow, Montanaro, 2012], also relying on verifier separability.

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Results:	Error Red	uction			

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- For even i > 0:
  - pure  $Q\Sigma_i(c,s) \subseteq pure Q\Sigma_{i_{OUT}}^{SEP}(1/np(n)^2, 1/e^n)$
  - pure  $Q\Pi_i(c,s) \subseteq pure Q\Pi_i^{SEP}(1-1/e^n, 1-1/np(n)^2)$
- Solution For odd i > 0:
  - pure  $Q\Sigma_i(c,s) \subseteq pure Q\Sigma_i^{SEP}(1-1/e^n, 1-1/np(n)^2)$
  - ② pureQ $\Pi_i(c,s)$  ⊆ pureQ $\Pi_i^{\text{SEP}}(1/np(n)^2, 1/e^n)$
- **Significance:** One-sided error reduction for QMA(2) [Aaronson, Beigi, Drucker, Fefferman, Shor, 2008] is the first step in the two-sided error reduction [Harrow, Montanaro, 2012], also relying on verifier separability.
  - Note,  $QMA(2) \subseteq Q\Sigma_3 \subseteq pureQ\Sigma_3$ .

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Results: U	pper & Lo	wer Bounds			

#### Lemma

For all even  $k \ge 2$ ,  $QC\Pi_k \subseteq pureQ\Pi_k$ .

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#### Lemma

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For all even k \ge 2, QC\Pi_k \subseteq pureQ\Pi_k.
```

#### Theorem

 $QCPH \subseteq pureQPH \subseteq EXP^{PP}$ 

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#### Lemma

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For all even k \ge 2, QC\Pi_k \subseteq pureQ\Pi_k.
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#### Theorem

 $QCPH \subseteq pureQPH \subseteq EXP^{PP}$ 

• Concurrent work: QCPH = DistributionQCPH ⊆ QPH, at the cost of constant factor blowup in level [Grewal, Yirka, 2024].

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#### Lemma

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For all even k \ge 2, QC\Pi_k \subseteq pureQ\Pi_k.
```

#### Theorem

 $QCPH \subseteq pureQPH \subseteq EXP^{PP}$ 

- Concurrent work: QCPH = DistributionQCPH ⊆ QPH, at the cost of constant factor blowup in level [Grewal, Yirka, 2024].
- Containment in EXP<sup>PP</sup> follows from Toda's theorem:

$$pureQ\Sigma_i \subseteq NEXP^{NP^{i-1}} \subseteq EXP^{NP^i} = EXP^{PP} = EXP^{PP}$$

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### Introduction

#### 2 Results



#### Error Reduction

#### 5 Lower Bounds



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Collapse theorem							

#### Collapse theorem for QCPH

If for any  $k \ge 1$ ,  $QC\Pi_k = QC\Sigma_k$ , then  $QCPH = QC\Sigma_k$ .

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Collapse theorem						

#### Collapse theorem for QCPH

If for any 
$$k \ge 1$$
,  $QC\Pi_k = QC\Sigma_k$ , then  $QCPH = QC\Sigma_k$ .

#### Lemma

If for any  $k \ge 1$ ,  $QC\Sigma_k = QC\Pi_k$ , then for all  $i \ge k$ ,  $QC\Sigma_i = QC\Pi_i = QC\Sigma_k$ .
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Lemma					
If for an	$hy \ k \ge 1, \ QC$	$\Sigma_k = \mathrm{QC}\Pi_k$ , then for all	$ll \ i \ge k, \ QC\Sigma_i = Q$	$C\Pi_i = QC\Sigma_k.$	
Proof					

Proof by induction: Assume  $QC\Sigma_i = QC\Pi_i = QC\Sigma_k$  for all  $j \in \{k, ..., i-1\}$ .

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Lem	na				
If for	$r$ any $k \ge 1$ , QCS	$C_k = QC\Pi_k$ , then for all	$ll \ i \ge k, \ QC\Sigma_i = Q$	$C\Pi_i = QC\Sigma_k.$	
Proc	f.				

Proof by induction: Assume  $QC\Sigma_j = QC\Pi_j = QC\Sigma_k$  for all  $j \in \{k, ..., i-1\}$ . Let  $L = (L_{yes}, L_{no}) \in QC\Sigma_i$  with verifier  $V_n$ . Define  $L' = (L'_{yes}, L'_{no})$ :

ntrod 000	uction Ro	esults 000	Quantum Karp-Lipton 0●0	Error Reduction 0000	Lower Bounds 00	Conclusion 00
	Lemma					
	If for any $k \ge$	$\pm 1, QC\Sigma_k = 0$	$QC\Pi_k$ , then for all $i \ge 1$	$k, QC\Sigma_i = QC\Pi_i$	$= \mathrm{QC}\Sigma_k.$	
	Proof.					
	Proof by indu	uction: Assum	ne $QC\Sigma_j = QC\Pi_j = Q$	$C\Sigma_k$ for all $j \in \{k, $	, <i>i</i> – 1}. Let	
	$L = (L_{\rm yes}, L_{\rm no})$	$) \in \mathrm{QC}\Sigma_i$ wit	h verifier $V_n$ . Define	$L' = (L'_{yes}, L'_{no})$ :		
	$L'_{y}$	$_{\mathrm{es}}=\left\{ (x,y_{1})\ \right $	$\forall y_2 \exists y_3 \cdots Q_i y_i \colon \Pr[\forall$	$V_n(x, y_1, \dots, y_i) = 1$	$1] \ge 2/3 \big\}$	
	$L_{ m r}^{\prime}$	$y'_{no} = \{(x, y_1) \mid$	$\exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \Pr[$	$V_n(x, y_1, \dots, y_i) = 1$	$\lfloor \rfloor \le 1/3 \bigr\}$	

roduction 00	Results 0000	Quantum Karp-Lipton õ●0	Error Reduction	Lower Bounds 00	Conclus 00
Lemma					
If for an	$hy \ k \ge 1, \ QCY$	$E_k = QC\Pi_k$ , then for a	$ll \ i \ge k, \ QC\Sigma_i = Q$	$C\Pi_i = QC\Sigma_k.$	
Proof.					
Proof b	y induction: /	Assume $QC\Sigma_j = QC\Pi_j$	$= QC\Sigma_k$ for all $j \in$	$\in \{k, \ldots, i-1\}$ . Let	et
$L = (L_{yo})$	$_{\mathrm{es}}, L_{\mathrm{no}}) \in \mathrm{QCZ}$	$\Sigma_i$ with verifier $V_n$ . De	fine $L' = (L'_{yes}, L'_{no})$	):	
	$L'_{\rm yes} = \big\{(x$	$(y_1) \mid \forall y_2 \exists y_3 \cdots Q_i y_i :$	$\Pr[V_n(x, y_1, \dots, y_n]]$	$(i_{i}) = 1] \ge 2/3$	
	$L'_{\rm no} = \left\{ (x_{\rm no}) \in X \right\}$	$(y_1) \mid \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i:$	$\Pr[V_n(x, y_1, \dots, y_n]]$	$(i) = 1] \le 1/3$	
$L' \in \mathbf{Q}$	$QC\Pi_{i-1} = QCZ$	$\Sigma_{i-1}$ . Thus there exists	$V'_n$ s.t.		
		$(x,y_1)\in L'_{\rm yes} \Rightarrow \exists y_2 \forall$	$y_3 \cdots \overline{Q_i} y_i$ : $\Pr[V'_n($	$[x, y_1, \ldots, y_i) = 1]$	$\geq 2/3$
		$(x,y_1)\in L'_{\rm no} \Rightarrow \forall y_2 \exists$	$y_3 \cdots Q_i y_i$ : $\Pr[V'_n($	$[x, y_1, \dots, y_i) = 1]$	$\leq 1/3$

rod 00	roduction Results Quantum Karp-Lipton Error Reduction NO 0000 0€0 0000	n Lower Bounds Conclus 00 00
	Lemma	
	If for any $k \ge 1$ , $QC\Sigma_k = QC\Pi_k$ , then for all $i \ge k$ , $QC\Sigma_k$	$_{i} = \mathrm{QC}\Pi_{i} = \mathrm{QC}\Sigma_{k}.$
	Proof.	
	Proof by induction: Assume $QC\Sigma_j = QC\Pi_j = QC\Sigma_k$ for $L = (L_{yes}, L_{no}) \in QC\Sigma_i$ with verifier $V_n$ . Define $L' = (L'_{yes}, L_{no})$	all $j \in \{k, \dots, i-1\}$ . Let $_{s}, L'_{no}$ ):
	$L'_{\text{yes}} = \left\{ (x, y_1) \mid \forall y_2 \exists y_3 \cdots Q_i y_i \colon \Pr[V_n(x, y_1, y_1)] \right\}$	, $y_i$ ) = 1] ≥ 2/3}
	$L_{\rm no}' = \left\{ (x, y_1) \mid \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \Pr[V_n(x, y_1, y_1)] \right\}$	, $y_i$ ) = 1] $\leq 1/3$
	$L' \in QC\Pi_{i-1} = QC\Sigma_{i-1}$ . Thus there exists $V'_n$ s.t.	
	$(x, y_1) \in L'_{\text{yes}} \Rightarrow \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon H$	$\Pr[V'_n(x, y_1, \dots, y_i) = 1] \ge 2/3$
	$(x, y_1) \in L'_{\mathrm{no}} \Rightarrow \forall y_2 \exists y_3 \cdots Q_i y_i \colon \mathbb{H}$	$\Pr[V'_n(x, y_1, \dots, y_i) = 1] \le 1/3$
	$x \in L_{\text{yes}} \Rightarrow \exists y_1 \colon (x, y_1) \in L'_{\text{yes}} \Rightarrow \exists y_1 \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \mathbb{F}$	$\Pr[V'_n(x, y_1, \dots, y_i) = 1] \ge 2/3$
	$x \in L_{\rm no} \Rightarrow \forall y_1 \colon (x, y_1) \in L'_{\rm no} \Rightarrow \forall y_1 \forall y_2 \exists y_3 \cdots Q_i y_i \colon \mathbb{R}$	$\Pr[V'_n(x, y_1, \dots, y_i) = 1] \le 1/3$

rod O	uction Results Quantum Karp-Lipton Error Reduction Lower Bounds 0000 000 000 000 000 00	00
	Lemma	
	If for any $k \ge 1$ , $QC\Sigma_k = QC\Pi_k$ , then for all $i \ge k$ , $QC\Sigma_i = QC\Pi_i = QC\Sigma_k$ .	
	Proof.	
	Proof by induction: Assume $QC\Sigma_j = QC\Pi_j = QC\Sigma_k$ for all $j \in \{k,, i-1\}$ . Let $L = (L_{yes}, L_{no}) \in QC\Sigma_i$ with verifier $V_n$ . Define $L' = (L'_{yes}, L'_{no})$ :	
	$L'_{\text{yes}} = \left\{ (x, y_1) \mid \forall y_2 \exists y_3 \cdots Q_i y_i \colon \Pr[V_n(x, y_1, \dots, y_i) = 1] \ge 2/3 \right\}$	
	$L_{\rm no}' = \left\{ (x, y_1) \mid \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \Pr[V_n(x, y_1, \dots, y_i) = 1] \le 1/3 \right\}$	
	$L' \in QC\Pi_{i-1} = QC\Sigma_{i-1}$ . Thus there exists $V'_n$ s.t.	
	$(x,y_1) \in L'_{\mathrm{yes}} \Rightarrow \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \Pr[V'_n(x,y_1,\ldots,y_i) = 1] \ge 2/2$	3
	$(x,y_1)\in L_{\rm no}' \Rightarrow \forall y_2 \exists y_3 \cdots Q_i y_i \colon \Pr[V_n'(x,y_1,\ldots,y_i)=1] \leq 1/2$	3
	$x \in L_{\mathrm{yes}} \Rightarrow \exists y_1 \colon (x, y_1) \in L_{\mathrm{yes}}' \Rightarrow \exists y_1 \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \colon \Pr[V_n'(x, y_1, \dots, y_i) = 1] \ge 2/2$	3
	$x \in L_{\rm no} \Rightarrow \forall y_1 \colon (x, y_1) \in L'_{\rm no} \Rightarrow \forall y_1 \forall y_2 \exists y_3 \cdots Q_i y_i \colon \Pr[V'_n(x, y_1, \dots, y_i) = 1] \le 1/2$	3
	Hence, $L \in QC\Sigma_{i-1}$ .	

Introduction 000	Results 0000	Quantum Karp-Lipton õo●	Error Reduction	Lower Bounds 00	Conclusio 00
Karp-L	ipton theorem	for QCPH			
If QCM	$A \subseteq BQP_{/\mathrm{mpo}}$	$_{\rm ly}$ , then QCPH = QC $\Sigma$	$C_2 = QC\Pi_2.$		
Proof s	ketch.				
• Let I	$L = (L_{\rm yes}, L_{\rm no})$	$\in \mathrm{QC}\Pi_2$ with verifier	$V_n$ and negligible	error $\epsilon$ .	

ntrodu 200	ction Results 0000	Quantum Karp-Lipton 00●	Error Reduction	Lower Bounds 00	Conclusio OO
	Karp-Lipton theore	m for QCPH			
	$\mathit{If}  QCMA \subseteq BQP_{/m}$	$_{\rm poly'}$ then QCPH = QCX	$\Sigma_2 = QC\Pi_2.$		
l	Proof sketch.				
• Let $L = (L_{\text{ves}}, L_{\text{no}}) \in \text{QC}\Pi_2$ with verifier $V_n$ and negligible error $\epsilon$ .					
	• Need to show <i>L</i>	$\in QC\Sigma_2$ . Define verifier	$V'_n(x, C, y_1) := V_n$	$(x, y_1, C(x, y_1))$ :	
	• $V_n'$ takes in	quantum circuit C, com	putes $y_2 = C(x, y_1)$ ,	and runs $V_n(x, y)$	$_{1}, y_{2}).$

ntrodu 000	ction Results 0000	Quantum Karp-Lipton 00●	Error Reduction 0000	Lower Bounds 00	Conclusio 00	
	Karp-Lipton theorer	n for QCPH				
	$\mathit{If}  QCMA \subseteq BQP_{/mp}$	$_{ooly'}$ then QCPH = QC $\Sigma$	$L_2 = QC\Pi_2.$			
	Proof sketch.					
	• Let $L = (L_{\text{ves}}, L_{\text{no}}) \in \text{QC}\Pi_2$ with verifier $V_n$ and negligible error $\epsilon$ .					
	• Need to show $L \in$	$QC\Sigma_2$ . Define verifier	$V'_n(x, C, y_1) := V_n$	$(x, y_1, C(x, y_1))$ :		
	• If $x \in L_{no}$ , then $\exists y \in C \exists y_1 \colon \Pr[V'_n(x)]$	$ \begin{aligned} y_1 \forall y_2 \colon & \Pr[V_n(x, y_1, y_2) \\ & C, y_1) = 1] \leq \epsilon. \end{aligned} $	$= 1 ] \leq \epsilon$ . Thus,			

ntroduc 000	tion Results 0000	Quantum Karp-Lipton 00●	Error Reduction 0000	Lower Bounds 00	Conclusic 00		
	Karp-Lipton theorem	for QCPH					
If QCMA $\subseteq$ BQP <sub>/mpoly</sub> , then QCPH = QC $\Sigma_2$ = QC $\Pi_2$ .							
	Proof sketch.						
	• Let $L = (L_{\text{yes}}, L_{\text{no}})$	$\in \mathrm{QC}\Pi_2$ with verifier	V <sub>n</sub> and negligible	error $\epsilon$ .			
	• Need to show $L \in$	$QC\Sigma_2$ . Define verifier	$V'_n(x, C, y_1) := V_n$	$(x, y_1, C(x, y_1))$ :			
	• If $x \in L_{no}$ , then $\exists y \\ \forall C \exists y_1 \colon \Pr[V'_n(x, y)]$	$ \begin{aligned} & y_1 \forall y_2 \colon \Pr[V_n(x, y_1, y_2) \\ & C, y_1) = 1] \leq \epsilon. \end{aligned} $	$= 1 ] \leq \epsilon$ . Thus,				

• If  $x \in L_{yes}$ , given  $y_1$ , the problem of finding  $y_2$  s.t.  $V_n(x, y_1, y_2)$  accepts, is in QCMA.

trodu 00	uction Results 0 0000 0	Quantum Karp-Lipton 00●	Error Reduction 0000	Lower Bounds 00	Conclusio 00
	Karp-Lipton theorem for Q	СРН			
	If $QCMA \subseteq BQP_{/mpoly'}$ the	en  QCPH = QCM	$\Sigma_2 = QC\Pi_2.$		
ļ	Proof sketch.				
	• Let $L = (L_{yes}, L_{no}) \in QC$	$\Pi_2$ with verifier	$V_n$ and negligible	error $\epsilon$ .	
	• Need to show $L \in QC\Sigma_2$	. Define verifier	$V'_n(x, C, y_1) := V_n$	$(x, y_1, C(x, y_1))$ :	
	• If $x \in L_{no}$ , then $\exists y_1 \forall y_2$ : $\forall C \exists y_1$ : $\Pr[V'_n(x, C, y_1)]$	$\Pr[V_n(x, y_1, y_2)] = 1] \le \epsilon.$	$)=1]\leq\epsilon.$ Thus,		
	• If $x \in L_{yes}$ , given $y_1$ , the	problem of findi	ng $y_2$ s.t. $V_n(x, y_1, y_2)$	$y_2$ ) accepts, is in (	QCMA.
	• Use randomized reduction	on from QCMA t	o UQCMA [Aharon	ov, Ben-Or, Brand	lão,

Sattah, 2022] to compute instance  $\phi_{(x,y_1)}$  with unique witness. By QCMA  $\subseteq$  BQP<sub>/mpolv</sub>,  $\exists$  circuit  $\tilde{C}$  deciding  $\phi_{(x,y_1)}$ .

troduction 00	Results 0000	Quantum Karp-Lipton õõ●	Error Reduction 0000	Lower Bounds 00	Conclusio 00
Karp-Li	pton theorem	for QCPH			
If QCM	$A \subseteq BQP_{/mpd}$	$_{\rm oby}$ , then QCPH = QC $\Sigma$	$L_2 = QC\Pi_2.$		
Proof sl	ketch.				
• Let L	$= (L_{\rm ves}, L_{\rm no})$	$\in \mathrm{QC}\Pi_2$ with verifier V	$V_n$ and negligible	error $\epsilon$ .	
• Need	to show $L \in$	$QC\Sigma_2$ . Define verifier	$V'_n(x, C, y_1) := V_n$	$(x, y_1, C(x, y_1))$ :	
• If $x \in \forall C \exists y$	$L_{no}$ , then $\exists y_1$ : $\Pr[V'_n(x, y_n)]$	$ \begin{aligned} &                                  $	$= 1 ] \leq \epsilon$ . Thus,		
• If $x \in$	$L_{\rm ves}$ , given y	1, the problem of findir	$y_{2}$ s.t. $V_{n}(x, y_{1}, y_{2})$	$y_2$ ) accepts, is in (	QCMA.

- Use randomized reduction from QCMA to UQCMA [ABBS22] to compute instance  $\phi_{(x,y_1)}$  with unique witness. By QCMA  $\subseteq$  BQP<sub>/mpoly</sub>,  $\exists$  circuit  $\tilde{C}$  deciding  $\phi_{(x,y_1)}$ .
- Apply search-to-decision reduction of [Irani, Natarajan, Nirkhe, Rao, Yuen, 2022] to construct circuit *C* that finds the unique witness for  $\phi_{(x,y_1)}$ :

$$\mathrm{Pr}_{y_2\leftarrow C(x,y_1)}[V_n(x,y_1,y_2)=1]\geq 1/\operatorname{poly}.$$

ntrodu 100	uction Results Quantum 0000 00●	Karp-Lipton Error 0000	Reduction	Lower Bounds 00	Conclusio 00
	Karp-Lipton theorem for QCPH				
	If $QCMA \subseteq BQP_{/mpoly'}$ then $QCMA \subseteq BQP_{/mpoly'}$	$CPH = QC\Sigma_2 = QCI$	Т <sub>2</sub> .		
ļ	Proof sketch.				
	• Let $L = (L_{\text{yes}}, L_{\text{no}}) \in \text{QC}\Pi_2$ w	ith verifier $V_n$ and m	egligible erro	r <i>ε</i> .	
	• Need to show $L \in QC\Sigma_2$ . De	fine verifier $V'_n(x, C,$	$(y_1) := V_n(x, y)$	$(y_1, C(x, y_1))$ :	
	• If $x \in L_{no}$ , then $\exists y_1 \forall y_2$ : $\Pr[\forall \forall C \exists y_1 : \Pr[V'_n(x, C, y_1) = 1]$	$V_n(x, y_1, y_2) = 1] \le \epsilon$ $\le \epsilon.$	e. Thus,		
	• If $x \in L_{yes}$ , given $y_1$ , the prob	em of finding $y_2$ s.t.	$V_n(x, y_1, y_2)$ is	accepts, is in QCN	ЛA.

- Use randomized reduction from QCMA to UQCMA [ABBS22] to compute instance  $\phi_{(x,y_1)}$  with unique witness. By QCMA  $\subseteq$  BQP<sub>/mpoly</sub>,  $\exists$  circuit  $\tilde{C}$  deciding  $\phi_{(x,y_1)}$ .
- Apply search-to-decision reduction of [INNRY22] to construct circuit *C* that finds the unique witness for  $\phi_{(x,y_1)}$ :  $\Pr_{y_2 \leftarrow C(x,y_1)}[V_n(x, y_1, y_2) = 1] \ge 1/$  poly.
- $\exists C \forall y_1 \colon \Pr[V'_n(x, C, y_1) = 1] \ge 1/\text{ poly.}$

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## Introduction



## 3 Quantum Karp-Lipton







Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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Asymmetric	Product T	est (APT)			

## Recall, for $L \in \text{pureQ}\Pi_i$ (*i* even), there exist c, s with $c - s \ge 1/\text{ poly s.t.}$

# Introduction Results Quantum Karp-Lipton Error Reduction Lower Bounds Conclusion Asymmetric Product Test (APT)

Recall, for  $L \in \text{pureQ}\Pi_i$  (*i* even), there exist c, s with  $c - s \ge 1/\text{ poly s.t.}$ •  $x \in L_{\text{yes}} \Rightarrow \forall |\psi_1\rangle \exists |\psi_2\rangle \cdots \exists |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \ge c$ •  $x \in L_{\text{no}} \Rightarrow \exists |\psi_1\rangle \forall |\psi_2\rangle \cdots \forall |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \le s$ 

# Introduction<br/>OCOResults<br/>OCOQuantum Karp-Lipton<br/>OCOError Reduction<br/>OCOLower Bounds<br/>OCOConclusion<br/>OCOAsymmetric Product Test (APT)

Recall, for  $L \in \text{pureQ}\Pi_i$  (*i* even), there exist c, s with  $c - s \ge 1/\text{ poly s.t.}$ •  $x \in L_{\text{yes}} \Rightarrow \forall |\psi_1\rangle \exists |\psi_2\rangle \cdots \exists |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \ge 1 - 1/\exp$ •  $x \in L_{\text{no}} \Rightarrow \exists |\psi_1\rangle \forall |\psi_2\rangle \cdots \forall |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \le 1/\exp$ 

• Could do "majority voting" if we get poly(n) copies of each state.

## Introduction Results Quantum Karp-Lipton Error Reduction Lower Bounds Conclusion Asymmetric Product Test (APT)

Recall, for  $L \in \text{pure}Q\Pi_i$  (*i* even), there exist c, s with  $c - s \ge 1/\text{ poly s.t.}$ 

- $x \in L_{\text{yes}} \Rightarrow \forall |\psi_1\rangle \exists |\psi_2\rangle \cdots \exists |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \ge 1 1/\exp(|\psi_1\rangle |\psi_1\rangle)$
- $x \in L_{no} \Rightarrow \exists |\psi_1\rangle \forall |\psi_2\rangle \cdots \forall |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \le 1 1/\text{ poly}$
- Could do "majority voting" if we get poly(n) copies of each state.
- If the last proof  $|\psi_i\rangle$  contains copies of all previous states, then we can still do majority voting. Verify this with the APT!

## Introduction Results Quantum Karp-Lipton Error Reduction Lower Bounds Conclusion Asymmetric Product Test (APT)

Recall, for  $L \in \text{pure}Q\Pi_i$  (*i* even), there exist c, s with  $c - s \ge 1/\text{ poly s.t.}$ 

- $x \in L_{\text{yes}} \Rightarrow \forall |\psi_1\rangle \exists |\psi_2\rangle \cdots \exists |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \ge 1 1/\exp(|\psi_1\rangle |\psi_1\rangle)$
- $x \in L_{no} \Rightarrow \exists |\psi_1\rangle \forall |\psi_2\rangle \cdots \forall |\psi_i\rangle$ :  $\Pr[V(|x, \psi_1, \dots, \psi_i\rangle) = 1] \le 1 1/\text{ poly}$
- Could do "majority voting" if we get poly(n) copies of each state.
- If the last proof  $|\psi_i\rangle$  contains copies of all previous states, then we can still do majority voting. Verify this with the APT!
- We construct new "asymmetric version" of the product test [Mintert, Kuś, Buchleitner, 2005] [Harrow, Montanaro, 2012]

Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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APT					

- Register A:  $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$  with  $|\psi_i\rangle \in \mathbb{C}^{d_i}$
- Register *B*:  $|\phi\rangle \in \mathbb{C}^{(d:=d_1 \cdots d_n)^m}$  (*m* copies of  $|\psi\rangle$ )

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APT					

- Register A:  $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$  with  $|\psi_i\rangle \in \mathbb{C}^{d_i}$
- Register *B*:  $|\phi\rangle \in \mathbb{C}^{(d:=d_1\cdots d_n)^m}$  (*m* copies of  $|\psi\rangle$ )
- **②** Choose  $(i, j) \in [n] \times [m]$  uniformly at random.

Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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APT					

- Register A:  $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$  with  $|\psi_i\rangle \in \mathbb{C}^{d_i}$
- Register *B*:  $|\phi\rangle \in \mathbb{C}^{(d:=d_1 \cdots d_n)^m}$  (*m* copies of  $|\psi\rangle$ )
- ② Choose  $(i, j) \in [n] \times [m]$  uniformly at random.
- Run SWAP test between *i*-th register of A and its *j*-th copy in B.



Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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APT					

- Register A:  $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$  with  $|\psi_i\rangle \in \mathbb{C}^{d_i}$
- Register *B*:  $|\phi\rangle \in \mathbb{C}^{(d:=d_1 \cdots d_n)^m}$  (*m* copies of  $|\psi\rangle$ )
- Oboose  $(i, j) \in [n] \times [m]$  uniformly at random.
- Run SWAP test between *i*-th register of A and its *j*-th copy in B.



#### Lemma

Let 
$$|\phi\rangle_{BC} \in \mathbb{C}^{d^m} \otimes \mathbb{C}^{d'}$$
 for some  $d' > 0$  s.t.  

$$\max_{\substack{|\psi\rangle:=|\psi_1\rangle\otimes\cdots\otimes|\psi_n\rangle\in\mathbb{C}^d}} \langle \phi|_{BC}[(|\psi\rangle\langle\psi|^{\otimes m})_B\otimes I_C]|\phi\rangle_{BC} = 1 - \epsilon$$
for  $\epsilon \ge 0$ .

Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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APT					

- Register A:  $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$  with  $|\psi_i\rangle \in \mathbb{C}^{d_i}$
- Register *B*:  $|\phi\rangle \in \mathbb{C}^{(d:=d_1\cdots d_n)^m}$  (*m* copies of  $|\psi\rangle$ )
- ② Choose  $(i, j) \in [n] \times [m]$  uniformly at random.
- Run SWAP test between *i*-th register of A and its *j*-th copy in B.



#### Lemma

 $\begin{array}{l} \text{Let } |\phi\rangle_{BC} \in \mathbb{C}^{d^{m}} \otimes \mathbb{C}^{d'} \text{ for some } d' > 0 \text{ s.t.} \\ & \max_{|\psi\rangle:=|\psi_{1}\rangle\otimes\cdots\otimes|\psi_{n}\rangle\in\mathbb{C}^{d}} \quad \langle \phi|_{BC}[(|\psi\rangle\langle\psi|^{\otimes m})_{B}\otimes I_{C}]|\phi\rangle_{BC} = 1 - \epsilon \\ \text{for } \epsilon \geq 0. \text{ Then, } APT \text{ accepts } |\eta\rangle_{ABC} := |\psi\rangle_{A} \otimes |\phi\rangle_{BC} \text{ w.p. } \leq 1 - \epsilon/2mn \text{ for all } |\psi\rangle. \end{array}$ 

Introduction	Results	Quantum Karp-Lipton	Error Reduction	Lower Bounds	Conclusion
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One-sided	Error Re	eduction			

Let *i* be even, 
$$c - s \ge 1/p(n)$$
. Then  

$$pureQ\Pi_i(c, s) \subseteq pureQ\Pi_i^{SEP}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right).$$

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One-sided	Error R	eduction			

Let *i* be even, 
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. Then  

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• Let V be be a verifier for  $L \in \text{pure}Q\Pi_i(c, s)$  s.t.

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One-sided	Error Red	uction			

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. Then  
pure $Q\Pi_i(c, s) \subseteq pure Q\Pi_i^{SEP}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right)$ .

- Let V be be a verifier for  $L \in \text{pure}Q\Pi_i(c, s)$  s.t.
- Verifier  $V'((|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_A \otimes |\psi'_i\rangle_{BC}$  with  $|\psi'_i\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_B^{\otimes m} \otimes |\psi_i\rangle_C^{\otimes m}$  (for honest last prover). Do with probability 1/2:

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One-sided	Error Red	uction			

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  - Apply APT between *A* and *B*.

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One-sided	Error R	eduction			

Let *i* be even, 
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  - Apply APT between *A* and *B*.
  - **②** Run original verifier V verifier on all *m* copies of  $|\psi_1\rangle, \ldots, |\psi_i\rangle$  inside *B*, *C* and accept if at least (c + s)/2 runs accept.

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One-sided	Error Red	uction			

Let *i* be even, 
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- Verifier  $V'((|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_A \otimes |\psi'_i\rangle_{BC})$  with  $|\psi'_i\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_B^{\otimes m} \otimes |\psi_i\rangle_C^{\otimes m}$  (for honest last prover). Do with probability 1/2:
  - Apply APT between *A* and *B*.
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- Note that APT also bounds entanglement between *B*, *C*.

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Introduction



- Ouantum Karp-Lipton
- Error Reduction





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QCPH ⊆ p	oureQPH				

For all even  $k \ge 2$ ,  $QC\Pi_k \subseteq pureQ\Pi_k$ .

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QCPH ⊆ pι	ıreQPH				

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For all even k \ge 2, QC\Pi_k \subseteq pureQ\Pi_k.
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• Let V be be a verifier for  $L \in QC\Pi_k$  s.t.

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QCPH ⊆ pι	ureQPH				

For all even  $k \ge 2$ ,  $QC\Pi_k \subseteq pureQ\Pi_k$ .

• Let V be be a verifier for  $L \in QC\Pi_k$  s.t.

• 
$$x \in L_{\text{yes}} \Rightarrow \forall y_1 \cdots \exists y_k$$
:  $\Pr[V(x, y_1, \dots, y_k) = 1] \ge 1 - 1/\exp$   
•  $x \in L_{\text{no}} \Rightarrow \exists y_1 \cdots \forall y_k$ :  $\Pr[V(x, y_1, \dots, y_k) = 1] \le 1/\exp$ 

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$QCPH \subseteq p$	ureQPH				

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- Let V be be a verifier for  $L \in QC\Pi_k$  s.t.
  - $x \in L_{yes} \Rightarrow \forall y_1 \cdots \exists y_k$ :  $\Pr[V(x, y_1, \dots, y_k) = 1] \ge 1 1 / \exp$ •  $x \in L_{ro} \Rightarrow \exists y_1 \cdots \forall y_k$ :  $\Pr[V(x, y_1, \dots, y_k) = 1] \le 1 / \exp$
- Define pure  $Q\Pi_k$  verifier  $V'(|x\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle \otimes |\psi'_k\rangle$ . Last proof  $|\psi'_k\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle)_B^{\otimes m} \otimes |\psi_k\rangle_C$  (honest prover).

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QCPH ⊆ pι	ureQPH				

For all even  $k \ge 2$ ,  $QC\Pi_k \subseteq pureQ\Pi_k$ .

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- Define pure  $Q\Pi_k$  verifier  $V'(|x\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle \otimes |\psi'_k\rangle$ . Last proof  $|\psi'_k\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle)_B^{\otimes m} \otimes |\psi_k\rangle_C$  (honest prover). Do with probability 1/2: • Run APT between A (i.e.  $|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle$ ) and B.
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| QCPH ⊆ p     | ureQPH  |                     |                 |              |            |

## Lemma

For all even  $k \ge 2$ ,  $QC\Pi_k \subseteq pureQ\Pi_k$ .

- Let V be be a verifier for  $L \in QC\Pi_k$  s.t.
  - $x \in L_{\text{yes}} \Rightarrow \forall y_1 \cdots \exists y_k \colon \Pr[V(x, y_1, \dots, y_k) = 1] \ge 1 1 / \exp[V(x, y_1, \dots, y_k)]$
  - $x \in L_{no} \Rightarrow \exists y_1 \cdots \forall y_k \colon \Pr[V(x, y_1, \dots, y_k) = 1] \le 1/\exp(1)$
- Define pure  $Q\Pi_k$  verifier  $V'(|x\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle \otimes |\psi'_k\rangle$ . Last proof  $|\psi'_k\rangle = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle)^{\otimes m}_B \otimes |\psi_k\rangle_C$  (honest prover). Do with probability 1/2:
  - Run APT between A (i.e.  $|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle$ ) and B.
  - Measure all proofs in standard basis and denote the outcomes by  $y_1, \ldots, y_k$ and  $y_{i,j}$  for the *j*-th copy of *i*.

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QCPH ⊆ p	ureQPH				

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    - If there exists i, j s.t.  $y_i \neq y_{i,j}$ , reject if i is even ( $\exists$ ), accept if i is odd ( $\forall$ ).

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  - Measure all proofs in standard basis and denote the outcomes by  $y_1, \ldots, y_k$  and  $y_{i,j}$  for the *j*-th copy of *i*.
    - If there exists i, j s.t.  $y_i \neq y_{i,j}$ , reject if i is even ( $\exists$ ), accept if i is odd ( $\forall$ ).
    - Otherwise, run  $V(x, y_1, \dots, y_k)$ .

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# Introduction

# 2 Results

- 3 Quantum Karp-Lipton
- Interpretation
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- **5** Lower Bounds



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Open Que	estions				

• Toda's theorem for QCPH, i.e.,  $QCPH \subseteq P^{PP}$ ?

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  - [GSSSY22] shows QCPH  $\subseteq P^{PP^{PP}}$ .

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- Error reduction for QPH?

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- Two-sided error reduction for pureQPH?

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- Two-sided error reduction for pureQPH?
- Collapse theorem for QPH/pureQPH?

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- Improve pureQPH  $\subseteq$  EXP<sup>PP</sup>. Is pureQ $\Sigma_3 \subseteq$  NEXP?

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  - Even  $pureQ\Pi_2 \in NEXP$  remains open.

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