On the Pure Quantum Polynomial Hierarchy and Quantified Hamiltonian Complexity

2510.06522

Sabee Grewal
The University of Texas at Austin

Dorian Rudolph
Paderborn University and PhoQS

The Polynomial Hierarchy (PH)

PH is a hierarchy of complexity classes generalizing NP. We *strongly* believe the levels are distinct (e.g., $\mathsf{P} \neq \mathsf{NP}$). The levels of PH are denoted Σ_k, Π_k for $k \geq 0$:

$$\Sigma_0 = \Pi_0 = \mathsf{P}, \quad \Sigma_1 = \mathsf{NP}, \Pi_1 = \mathsf{coNP}, \quad \Sigma_2 = \mathsf{NP}^{\mathsf{NP}}, \Pi_2 = \mathsf{coNP}^{\mathsf{NP}}, \ldots$$

Game-theoretic interpration: $L \in \Sigma_k$ can equivalently be defined as a game:

- *Alice* and *Bob* exchange k messages with a poly-time *verifier* V
- Alice ($\exists$) wins if she can convince V that $x \in L$ (YES)
- Bob ($\forall$) wins if we can convince V that $x \notin L$ (NO)
- In the YES (resp. NO) case Alice (resp. Bob) always has a winning strategy

Formally: $\quad x \in L \iff \exists y_1 \forall y_2 \cdots \exists y_{k-1} \forall y_k : V(x, y_1, \ldots, y_k) = 1$

How to define *Quantum* PH?

The oracle definition $\mathsf{QMA}^{\mathsf{QMA}^{\cdots}}$ seems unnatural: Oracle barriers don't allow joint measurements on quantum witnesses.

$\Longrightarrow$ **Use quantifier/game definition!** [GSS+22, AGKR24, GY24]

Aside: $\mathsf{QMA}^{\mathsf{QMA}^{\cdots}} \subseteq \mathsf{PSPACE}$, but QPH vs. NEXP remains open!

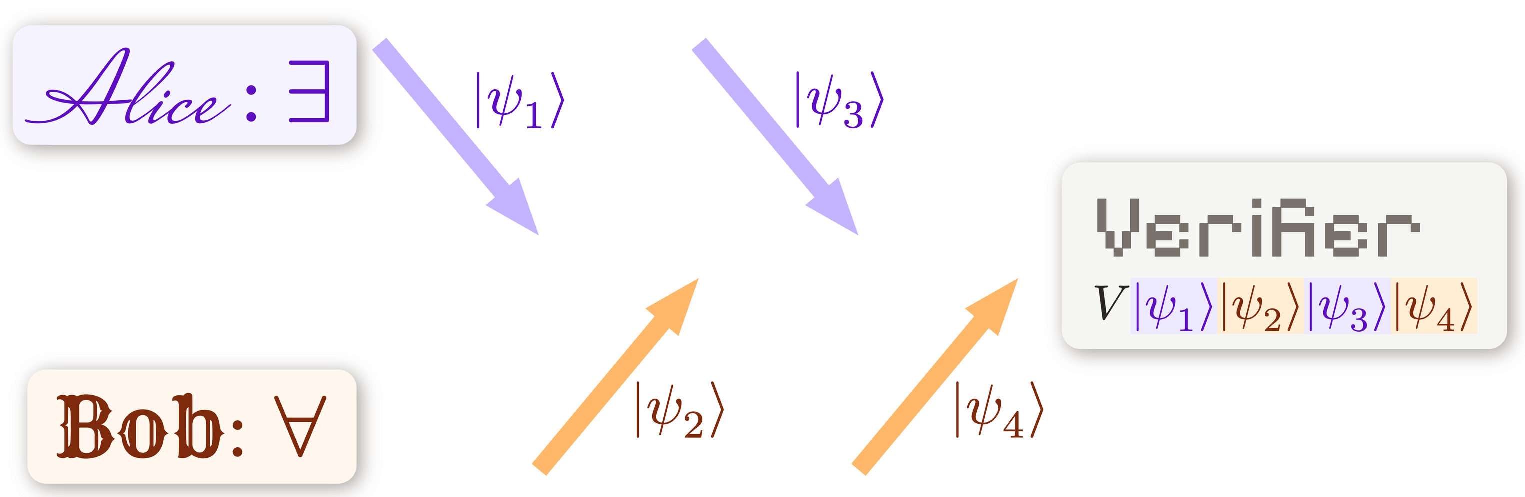

Figure 1. $\mathsf{pureQ}\Sigma_4$

Formally: Promise problem $L = (L_{\text{yes}}, L_{\text{no}}) \in \mathsf{Q}\Sigma_k$ (for even k) if

$$x \in L_{\text{yes}} \implies \exists \rho_1 \forall \rho_2 \cdots \exists \rho_{k-1} \forall \rho_k : \quad \Pr[V_x(\rho_1 \otimes \cdots \otimes \rho_k) = 1] \geq c$$
$$x \in L_{\text{no}} \implies \forall \rho_1 \exists \rho_2 \cdots \forall \rho_{k-1} \exists \rho_k : \quad \Pr[V_x(\rho_1 \otimes \cdots \otimes \rho_k) = 1] \leq s$$

For $c - s \geq \frac{1}{\text{poly}(n)}$. Then $\mathsf{QPH} := \bigcup_k \mathsf{Q}\Sigma_k$.

Alternative: Define pureQPH analogously with pure states $|\psi_1\rangle, \ldots, |\psi_k\rangle$.

Mixed vs. Pure

Which is the *"right"* definition: QPH or pureQPH?

- Trivial: $\mathsf{Q}\Sigma_k \subseteq \mathsf{pureQ}\Sigma_k$ by purification. $\Longrightarrow \mathsf{QPH} \subseteq \mathsf{pureQPH}$.
- But $\mathsf{pureQ}\Sigma_k$ *seems* more **powerful** than (mixed) $\mathsf{Q}\Sigma_k$!

Theorem 1. $\mathsf{QMA(2)} \subseteq \mathsf{pureQ}\Sigma_2 \subseteq \mathsf{Q}\Sigma_3 \subseteq \mathsf{NEXP}$

- *Definition* of QMA(2): Like QMA but with unentangled witness, i.e.

$$x \in L_{\text{yes}} \implies \exists |\psi_1\rangle \exists |\psi_2\rangle : \quad \Pr[V_x(|\psi_1\rangle \otimes |\psi_2\rangle) = 1] \geq \tfrac{2}{3}$$
$$x \in L_{\text{no}} \implies \forall |\psi_1\rangle \forall |\psi_2\rangle : \quad \Pr[V_x(|\psi_1\rangle \otimes |\psi_2\rangle) = 1] \leq \tfrac{1}{3}$$

- Evidence that $\mathsf{pureQ}\Sigma_2 \neq \mathsf{Q}\Sigma_2$ since $\mathsf{Q}\Sigma_2 \subseteq \mathsf{PSPACE}$ [GSS+22], but QMA(2) is conjectured to be much harder (might equal NEXP).

QMA(2) **as non-convex max-min optimization:** Approximate for POVM Π

$$\max_{|\psi\rangle} \min_{|\varphi\rangle} \operatorname{Tr}\big(\Pi(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)\big).$$

Amplification

- [HM13] shows amplification of QMA(2) to completeness $c = 1 - \frac{1}{\exp(n)}$ and soundness $s = \frac{1}{\exp(n)}$. Does not generalize to QPH due to "mixed vs. pure" issue and non-convexity.
- [AGKR24] gives *one-sided* amplification $c = 1 - \frac{1}{\exp}, s = 1 - \frac{1}{\text{poly}}$.
- **Amplification remained open** and thus definition required c, s parameters.

Theorem 2. $\mathsf{pureQ}\Sigma_k(c, s) \subseteq \mathsf{Q}\Sigma_{7k}\Big(1 - \frac{1}{\text{poly}(n)}, \frac{1}{\text{poly}(n)}\Big)$ for $c - s \geq \frac{1}{\text{poly}(n)}$.

Corollary. $\quad \mathsf{QPH} = \mathsf{pureQPH}$

Disentanglers with poly-size support

Proof idea for amplification: Use [JW24] dimension-independent disentangler from unentanglement, mapping $\rho_1 \otimes \rho_2$ to $\int |\psi\rangle\langle\psi|^{\otimes k} \, d\mu(\psi)$ with error $\frac{1}{\text{poly}(n)}$.

- **Challenge:** JW disentangler creates mixed state. Alice knows the correct response to each $|\psi\rangle$, but she cannot send all!
- **Solution:** Disentangler creating mixed state with poly-size support.

Lemma. Efficient quantum channel $\Gamma : \mathcal{D}(\mathcal{H}^{\otimes \ell}) \to \mathcal{D}(\mathcal{H}^{\otimes k})$, s.t.

$$\left\| \Gamma(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4) - \sum_{i=1}^{m} p_i |\eta_i\rangle\langle\eta_i|^{\otimes k} \right\|_1 \leq \delta,$$

where $\ell = \text{poly}(\delta^{-1}, k), m = O(\delta^{-2})$, and $|\eta_i\rangle = |\eta_{i,1}\rangle \otimes \cdots \otimes |\eta_{i,s}\rangle$.

- Now Alice can send tensor product of $|\eta_i, \psi_i\rangle$ for each of Bob's m $|\eta_i\rangle$'s.
- Verifier picks Alice's response via SWAP tests.

Quantified Hamiltonian problems

$\exists\forall$ *quantified local Hamiltonian problem* ($\exists\forall$-MLH): Distinguish
(YES) $\exists \rho \forall \sigma : \operatorname{Tr}(H(\rho \otimes \sigma)) \leq a$ and (NO) $\forall \rho \exists \sigma : \operatorname{Tr}(H(\rho \otimes \sigma)) \geq b$.

Proposition. $\exists\forall\text{-MLH} \in \mathsf{NP}^{\mathsf{QMA}} \cap \mathsf{coNP}^{\mathsf{QMA}}$.

- No joint measurements needed!

Complete problems for levels of pureQPH**:**

- Define *Pure-Sparse-Hamiltonian*-Σ_k (PSH-Σ_k) analogously to Σ_k, but with *sparse Hamiltonian* H as objective function, e.g., PSH-Σ_2:
  - (YES) $\exists |\psi\rangle \forall |\varphi\rangle : \operatorname{Tr}(H(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)) \leq a$
  - (NO) $\forall |\psi\rangle \exists |\varphi\rangle : \operatorname{Tr}(H(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)) \geq b$

Proposition. PSH-Σ_k/PSH-Π_k is $\mathsf{pureQ}\Sigma_k/\mathsf{pureQ}\Pi_k$-complete.

- MSH-$\Sigma_k \in \mathsf{Q}\Sigma_k$, but hardness seems tricky — SWAP test needs pure states!

Takeaway. pureQPH is the "right" definition of *Quantum* PH.

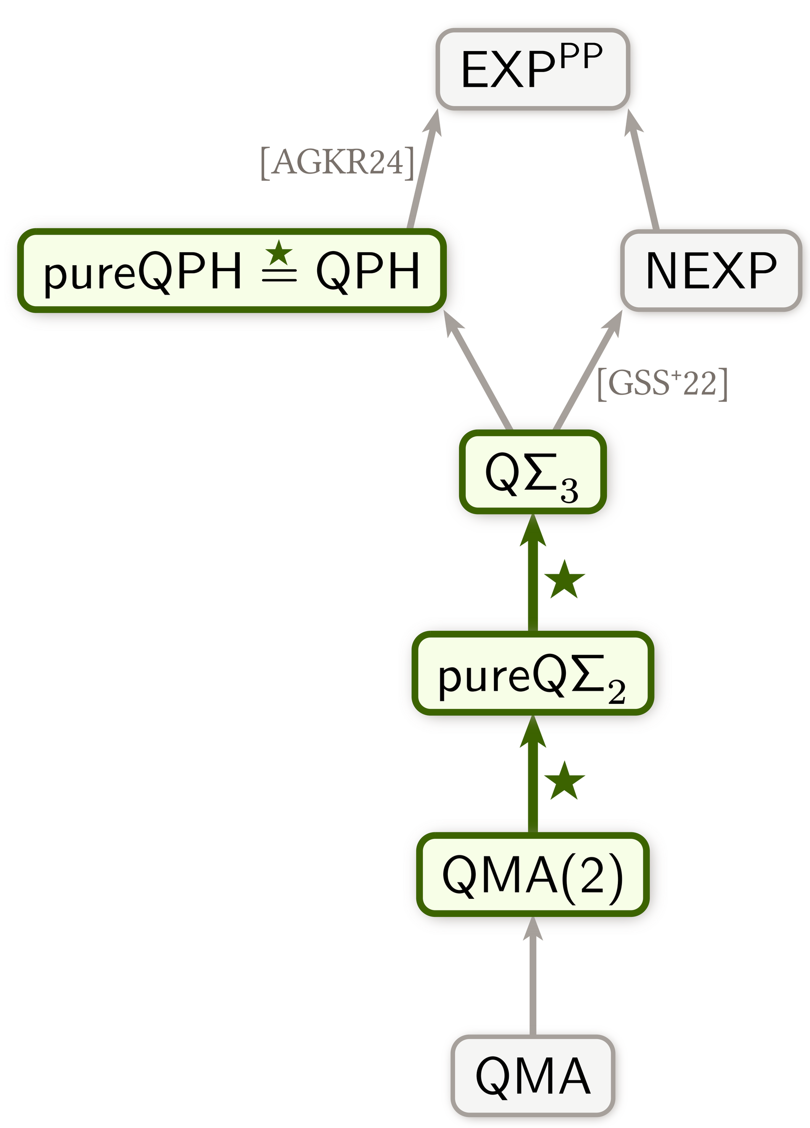

Figure 2. ★ = Our results.

Open questions

- Complete problems for other hierarchy variants?
  - Consider pure/local, mixed/local, or mixed/sparse Hamiltonian?
- Investigate $\mathsf{NP}^{\mathsf{QMA}}$ vs $\mathsf{QMA}^{\mathsf{QMA}}$.
  - $\exists\forall$-local Hamiltonian only needs quantumness in the oracle.
- Construct disentanglers with stronger guarantees.
  - Can the output always be close to a pure state?
- QPH vs. NEXP?

References

[AGKR24] Avantika Agarwal, Sevag Gharibian, Venkata Koppula, and Dorian Rudolph. "Quantum Polynomial Hierarchies: Karp-Lipton, Error Reduction, and Lower Bounds". In: *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*. 2024.

[GSS+22] Sevag Gharibian, Miklos Santha, Jamie Sikora, Aarthi Sundaram, and Justin Yirka. "Quantum Generalizations of the Polynomial Hierarchy with Applications to QMA(2)". In: *computational complexity* 31.2 (2022).

[GY24] Sabee Grewal and Justin Yirka. "The Entangled Quantum Polynomial Hierarchy Collapses". In: *39th Computational Complexity Conference (CCC 2024)*. 2024.

[HM13] Aram W. Harrow and Ashley Montanaro. "Testing Product States, Quantum Merlin-Arthur Games and Tensor Optimization". In: *Journal of the ACM* 60.1 (2013).

[JW24] Fernando Granha Jeronimo and Pei Wu. "Dimension Independent Disentanglers from Unentanglement and Applications". In: *39th Computational Complexity Conference (CCC 2024)*. 2024.