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Quantum 2-SAT on low dimensional systems is QMA_1-complete: Direct embeddings and black-box simulation

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Introduction

Definition (Quantum k-SAT). Given Hamiltonian $H = \sum_i H_i \in \mathbb{C}^{2^n \times 2^n}$ on n qubits with k-local projectors H_i, decide whether the H_i have a common nullspace.

Note: QSAT is like the local Hamiltonian problem, but with the additional promise that H is frustration-free in the YES-case.

$$k\text{-LH} \sim \text{MAX-}k\text{-SAT} \quad \& \quad k\text{-QSAT} \sim k\text{-SAT}$$

Complexity of k-LH is well understood (QMA-complete for $k=2$). For k-QSAT we know:

- 2-QSAT is in P [Bra06].
- 3-QSAT is QMA_1-complete [GN13] (QMA_1 is QMA with perfect completeness).

→ Complexity transition like in the classical setting. But in quantum, the study of *systems of higher local dimension* are also physically motivated; 2-QSAT becomes much harder:

Definition ((k,l)-QSAT). 2-QSAT, where each projector acts on a qu-k-it and a qu-l-it (i.e. $\mathbb{C}^k \otimes \mathbb{C}^l$).

- $(3,5)$-QSAT is QMA_1-complete [ER08].
- $(2,3)$-QSAT is NP-hard, classical $(2,3)$-SAT is NP-complete [Nag08].

Quantum SAT on a line of qu-d-its is also of great interest:

- Classical $1D$-SAT is in P for any local dimension.
- There exists a frustration free *qutrit* construction with *unique, entangled* ground state [BCMNS12].
  ⇒ First step to show that *entangled witness* can be encoded
- QMA_1-complete for $d=11$ [AGIK09; Nag08]

What is the smallest local dimension that can encode a QMA_1-hard problem?

Results

We improve the state of the art regarding (k,l)-QSAT:

Theorem. $(2,5)$-QSAT and $(3,4)$-QSAT is QMA_1-complete with soundness $\Omega(1/T^2)$.

- T denotes the number of gates in the original QMA_1-verifier embedded in the QSAT-instance.
- Soundness s requires that $\lambda_{\min}(H) \geq s$ in the NO-case.
- First demonstration that $(2,k)$-QSAT is QMA_1-hard.

Hardness of (k,l)-QSAT (new results bold red):

	2	3	4	5	6
2	∈ P	NP-H	NP-H	**QMA_1-C**	**QMA_1-C**
3		NP-H	**QMA_1-C**	QMA_1-C	QMA_1-C
4			**QMA_1-C**	QMA_1-C	QMA_1-C
5				QMA_1-C	QMA_1-C

Proof outline.

(1) *Circuit-to-Hamiltonian* embedding based on [GN13]: Requires a *clock Hamiltonian* whose nullspace is spanned by clock states, as well as (1-local) transition and selection projectors for clock states.
- Implement CNOT gates in $2D$ to reduce dimensions from $(3,5)$ [ER08] to $(3,4)$.

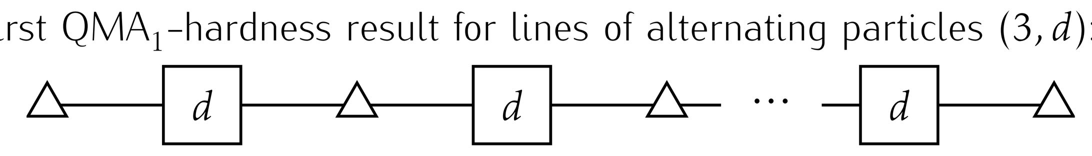
(2) Prove soundness via novel *Nullspace Connection Lemma*.

(3) Construct $(2,5)$ clock Hamiltonian by combining qu-5-its and qubits into logical qu-4-its and qutrits.

We further improve soundness of the $2D$-Hamiltonian [GN13] from $\Omega(1/T^6)$ to $\Omega(1/T^2)$ via a slight modification and analysis using *Unitary Labelled Graphs* [BCO17] and our Nullspace Connection Lemma.

Theorem. 3-QSAT is QMA_1-complete with soundness $\Omega(1/T^2)$.

First QMA_1-hardness result for lines of alternating particles $(3,d)$:



Theorem. $1D$-$(3,d)$-QSAT is QMA_1-complete for $d = 76176$.

- Black-box simulation of any $1D$-(d,d)-QSAT $1D$-$(3,d')$-QSAT with $d' \in O(d^4)$, preserving soundness in the NO-case (similar to *Hamiltonian simulation* [BH17; CMP18])
- Theorem follows from the QMA_1-hardness of $1D$-$(11,11)$-QSAT [Nag08].

Towards qubits on the line: Even $1D$-$(2,4)$-QSAT can have an entangled ground state.

Theorem. There exists a frustration-free $1D$-$(2,4)$-Hamiltonian with unique ground state entangled across all cuts.

2D Hamiltonian

Clock Hamiltonian H_{clock} with $\mathcal{N}(H_{\text{clock}}) = \text{Span}\{|C_1\rangle, \ldots, |C_n\rangle\}$

- *Transition projectors* $h_{i,i+1}(U)$ on $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock}}$ with

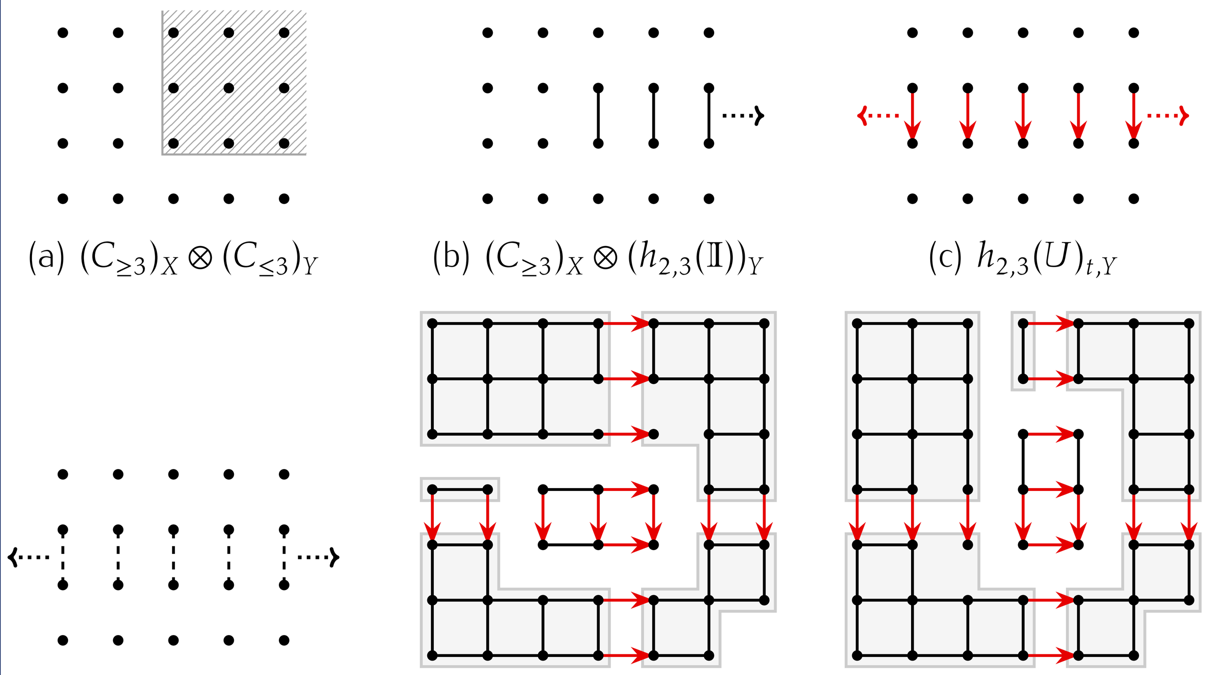
$$(\mathbb{I} \otimes \Pi_{\text{clock}})\, h_{i,i+1}(U)\, (I \otimes \Pi_{\text{clock}}) \sim (\mathbb{I} \otimes |C_i\rangle\langle C_i| + \mathbb{I} \otimes |C_{i+1}\rangle\langle C_{i+1}|) - (U^\dagger \otimes |C_i\rangle\langle C_{i+1}| + U \otimes |C_{i+1}\rangle\langle C_i|).$$

- *Selection projectors* $C_{\leq i}, C_{\geq i}$ on $\mathcal{H}_{\text{clock}}$ with

$$\Pi_{\text{clock}} C_{\geq i} \Pi_{\text{clock}} \sim \textstyle\sum_{j=i}^N |C_j\rangle\langle C_j|.$$

Theorem (informal, based on [GN13]). Any QMA_1-verifier can be reduced to a QSAT instance H on $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock},1} \otimes \mathcal{H}_{\text{clock},2}$ (Z,X,Y). H only has terms of the form $(H_{\text{clock}})_X + (H_{\text{clock}})_Y, C_{\leq x} \otimes C_{\geq y}, h_{i,i+1}(U)_{Z,X}, |0\rangle\langle 0|_Z \otimes h_{i,i+1}(\mathbb{I})_X$. Under certain commutation assumptions, soundness is $\Omega(\gamma(H_{\text{clock}})/N^2)$, for spectral gap $\gamma(\cdot)$.

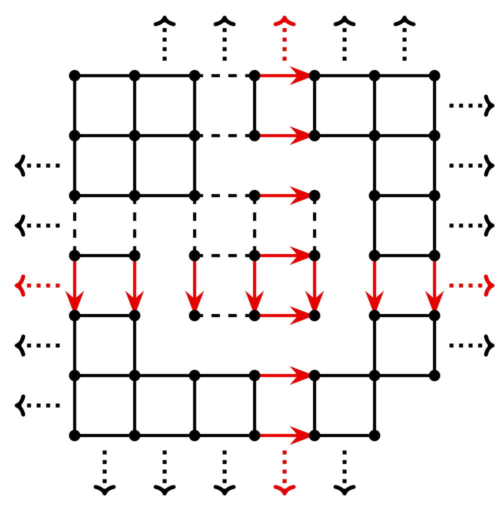
Gadgets of the $2D$-Hamiltonian on the grid of clock states (modified from [GN13]):

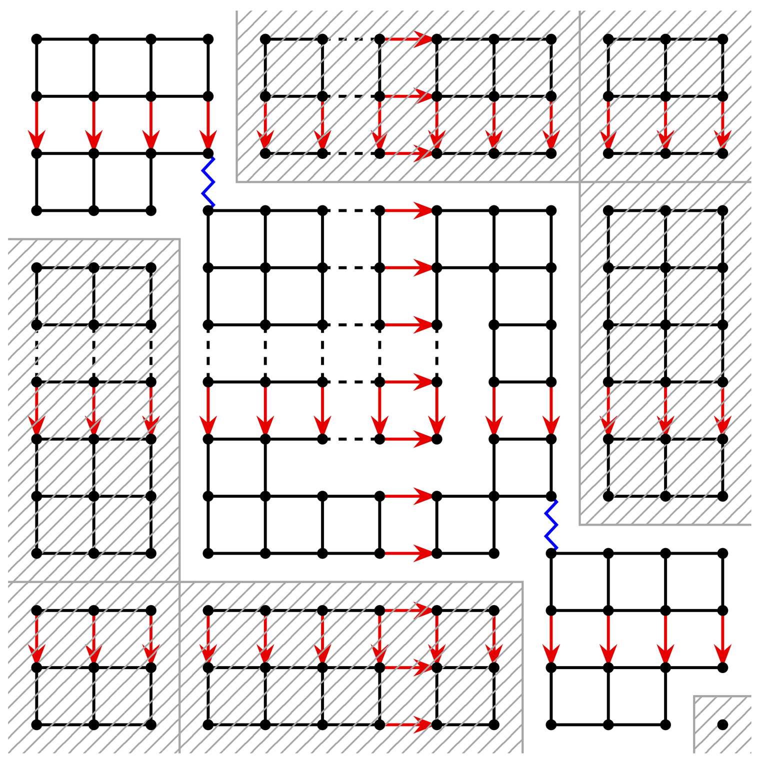


(a) $(C_{\geq 3})_X \otimes (C_{\leq 3})_Y$ (b) $(C_{\geq 3})_X \otimes (h_{2,3}(\mathbb{I}))_Y$ (c) $h_{2,3}(U)_{t,Y}$ (d) $|1\rangle\langle 1|_c \otimes (h_{2,3})_Y$ (e) CNOT on $|0\rangle_c$ (f) CNOT on $|1\rangle_c$

CNOT gadget (up to 1-local rotation):

- Transitions in X- and Y-direction implement non-commuting U_x, U_y.
- Cond. transitions route $|0\rangle$ through U_xU_y (e), and $|1\rangle$ through U_yU_x (f).
- Implements gate $V = |0\rangle\langle 0| \otimes U_xU_y + |1\rangle\langle 1| \otimes U_yU_x$ from top left to bottom right.
- "Center" of (e), (f) is penalized since $U_xU_y \neq U_yU_x$.





Embed circuit U_2VU_1 by chaining gadgets together (*Note:* separate gadget for 1-local gate):

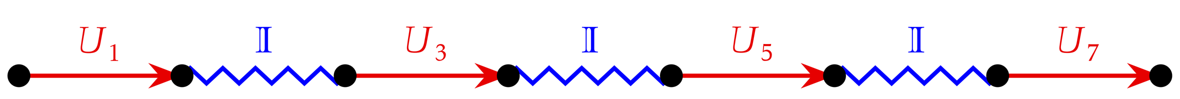
- Zig-zag edges represent $\mathbb{I}$-transitions connecting the gadgets.
- Each gadget embeds a single gate from TL to BR.
- Soundness follows from the next lemma:

Nullspace Connection Lemma (informal). Given circuit Hamiltonian on a graph of clock states:

- H_1 (gate gadgets): m circuit Hamiltonians on clock states K_i ($K_i \cap K_{j\neq i} = \emptyset$) implementing U_i from *input node* u_i to *output* v_i.
- H_2 (zig-zag edges): $\mathbb{I}$-transitions from v_i to u_{i+1}.

Then $H = H_1 + H_2$ embeds circuit $U_m \cdots U_1$ from u_1 to v_m with $\gamma(H) = \Omega(\gamma(H_1)/m^2)$.

Note: This lemma also applies to Kitaev's original circuit Hamiltonian and gives soundness without the random walk approach.



- The nullspace of red edges on their time steps is spanned by $|\psi\rangle \otimes |t-1\rangle + U_t|\psi\rangle \otimes |t\rangle$.
- Connect nullspaces to history state $\sum_{t=0}^7 U_t \cdots U_1|\psi\rangle|t\rangle$.
- Use H_{in} to set $|\psi\rangle = |0\rangle$ on $t=0$.

(2,5)-Clock

All that remains to show QMA_1-hardness of $(2,5)$-QSAT, is a $(2,5)$-clock with 1-local transition projectors $h_{i,i+1}$ and selection projectors $C_{\leq i}, C_{\geq i}$.

Idea: Use logical qudits with *indicator qubits* so that the qudit can be "selected" with a qubit projector. Example:

- Implement logical qutrit with qu-6-it and three qubits.
- $|\hat{1}\rangle \propto |1\rangle|000\rangle + |4\rangle|100\rangle$, $|\hat{2}\rangle \propto |2\rangle|000\rangle + |5\rangle|010\rangle$, $|\hat{3}\rangle \propto |3\rangle|000\rangle + |6\rangle|001\rangle$ (enforce with $(2,6)$-projectors)

Implement a logical $(3,4)$-clock.

- Qutrits can represent three states *unborn, alive, dead*, which seems necessary for a clock with 2-local transitions (see [ER08]).
- Qu-4-its can have two *alive* states, allowing for 1-local transitions.
- We need to limit use of indicator qubits compared to example since each increases the qu-d-it dimension.

Logical qutrit on qu-5-it (u,a,d,x,d') and two qubits:

$$|\mathrm{u}\rangle \propto |u,1,0\rangle + |x,0,0\rangle$$
$$|\mathrm{a}\rangle \propto |a,0,0\rangle + |x,1,0\rangle$$
$$|\mathrm{d}\rangle \propto |d,0,0\rangle + |d',0,1\rangle$$

Logical qu-4-it on qu-5-it (u,a_1,a_2,d,u') and one qubit:

$$|\mathrm{U}\rangle \propto |u,0\rangle + |u',1\rangle$$
$$|\mathrm{A}_1\rangle \propto |a_1,0\rangle$$
$$|\mathrm{A}_2\rangle \propto |a_2,0\rangle$$
$$|\mathrm{D}\rangle \propto |d,0\rangle$$

Key features: 1-local qubit projector suffices to

- identify $|\mathrm{U}\rangle$, $|\mathrm{d}\rangle$, "$|\mathrm{u}\rangle$ or $|\mathrm{a}\rangle$"
- transition from $|\mathrm{u}\rangle$ to $|\mathrm{a}\rangle$.

Clock states: $|C_1\rangle, \ldots, |C_N\rangle$ with

$$|C_{i-1}\rangle \begin{cases} \cdots \\ +\,|\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dA_1}\;\, \mathrm{uU}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \end{cases}$$
$$|C_i\rangle \begin{cases} |\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dA_2}\;\, \mathrm{uU}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \\ +\,|\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dD}\;\, \mathrm{aU}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \\ +\,|\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dD}\;\, \mathrm{dU}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \\ +\,|\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dD}\;\, \mathrm{dA_1}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \end{cases}$$
$$|C_{i+1}\rangle \begin{cases} |\mathrm{dD} \;\cdots\; \mathrm{dD}\;\, \mathrm{dD}\;\, \mathrm{dA_2}\;\, \mathrm{uU} \;\cdots\; \mathrm{uU}\rangle \\ +\,\cdots \end{cases}$$

We implement $|C_i\rangle \leftrightarrow |C_{i+1}\rangle$ transitions as $|a_1\rangle \leftrightarrow |a_2\rangle$.

Remark: We get a $(3,4)$-clock "for free".

We can enforce the clock space with a $(2,5)$-QSAT Hamiltonian.

Projectors (each 522 52 block above is labeled $\alpha\beta\gamma\,\delta\epsilon$):

$$h_{i,i+1}(U_Z) = \frac{1}{2}(\mathbb{I}_Z \otimes |a_1\rangle\langle a_1|_{\delta_i} + \mathbb{I}_Z \otimes |a_2\rangle\langle a_2|_{\delta_i} - U_Z^\dagger \otimes |a_1\rangle\langle a_2|_{\delta_i} - U_Z \otimes |a_2\rangle\langle a_1|_{\delta_i})$$

$$C^{(5)}_{\leq i} = |u\rangle\langle u|_{\delta_i} \qquad C^{(2)}_{\leq i} = |1\rangle\langle 1|_{\epsilon_i}$$
$$C^{(5)}_{\geq i} = |d\rangle\langle d|_{\alpha_i} \qquad C^{(2)}_{\geq i} = |1\rangle\langle 1|_{\gamma_i}$$

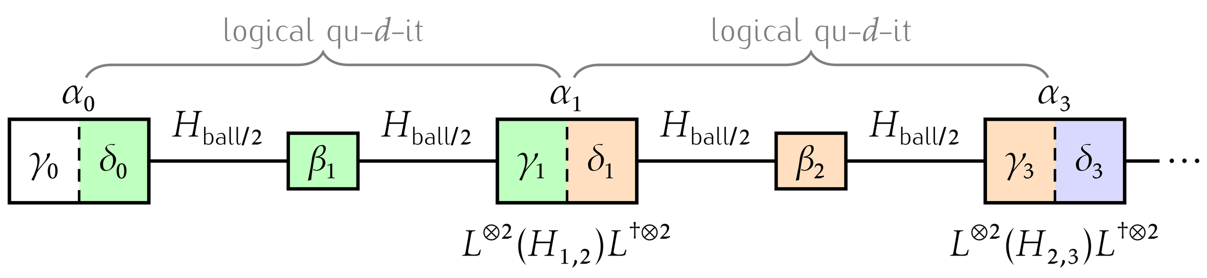
Note: We need qu-5-it and qubit variants of $C_{\leq i}, C_{\geq i}$ for bipartiteness.

1D-(3, d)-QSAT

Lemma (black-box simulation). Let $H = \sum_{i=1}^{n-1} H_{i,i+1}$ be a Hamiltonian on a line of n qu-d-its with $H_{i,i+1} \succeq 0$. There exists a poly-time computable H' on an alternating chain of $n+1$ qu-d'-its ($d' = O(d^4)$) and n qutrits, such that $\lambda_{\min}(H) = 0$ iff $\lambda_{\min}(H') = 0$ and $\gamma(H') = \Omega(\gamma(H)/\|H\|)$.

Idea: Logically split qu-d'-its into two qu-d''-its ($d'' = \sqrt{d'}$).

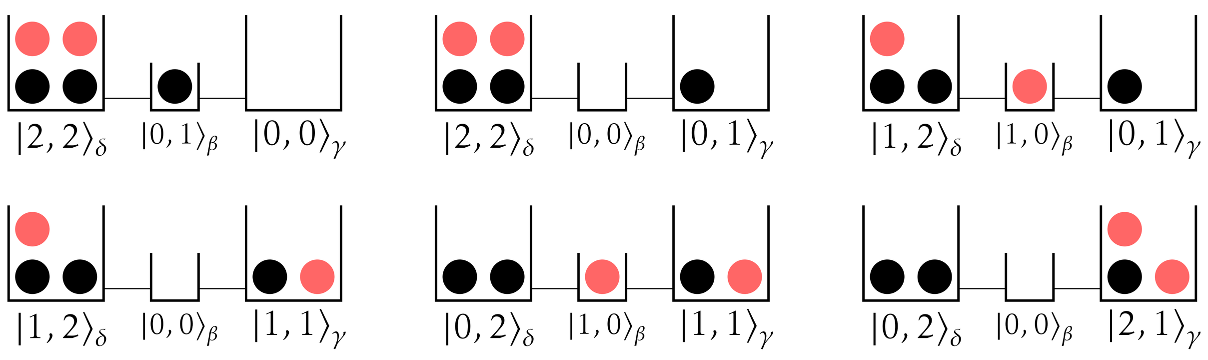
- Each $d'' \times 3 \times d''$ system logically represents a qu-d-it.
- "Move" the logical qu-d-it between the qu-d''-its so that $H_{i,i+1}$ can be simulated 1-locally on qu-d'-it α_i.



Logical qu-d-it (enforced with $H_{\text{ball}/2}$):

- Think of the $d'' \times 3 \times d''$ systems as bins able to hold *red* or *black* balls. Outer bins have capacity $2d$ and inner only 1.
- Write standard basis states as $|c_1, c_2\rangle$ (c_1 red and c_2 black balls).
- Transitions enforce superposition of all configurations with the same total number of red and black balls. E.g., $|2,1\rangle|0,0\rangle \leftrightarrow |1,1\rangle|1,0\rangle$.
- For $i \in [d]$, logical $|\hat{\imath}\rangle$ is superposition over all configurations with $c^*_{i,1} := 2(i-1)$ red and $(c^*_{i,2} := 2(d-i+1))$ black balls.
  ⇒ A full outer bin with even number of red balls determines $|\hat{\imath}\rangle$.
- In total $2d+1$ balls, so that if an outer bin is empty, the middle bin must be occupied. → Penalize the "all empty" configuration.

Some configurations of $|\hat{2}\rangle$ for $d=2$:



We can now *lift* the original Hamiltonian terms $H_{i,i+1}$ into the logical space via the isometry $L = \sum_{i=1}^d |c^*_{i,1}, c^*_{i,2}\rangle\langle i|$.

- Let $H' = H'_{\text{logical}} + H'_{\text{sim}}$, where $H'_{\text{sim}} = \sum_i (L \otimes L)(H_{i,i+1})(L \otimes L)^\dagger_{\alpha_i}$.
- Similar to Hamiltonian simulation, we obtain:

Lemma. $T^\dagger H' T = \frac{1}{9} H$ for isometry T.

Entangled 1D-(2, 4)-QSAT

We again use balls and bins, but now just a single color. *Large bins* have capacity 3 and small bins capacity 1. Configurations evolve with the following rules (also in reverse):

- First large bin always has at least one ball, and last at most 2.
- If large bin is nonempty, small bin to the right empty, add one ball to both: $|c,0\rangle \leftrightarrow |c+1,1\rangle$.
- If small bin is nonempty and large bin to the right empty, move ball: $|1,0\rangle \leftrightarrow |0,1\rangle$.

Uniform superposition of configurations ($l\ s \mid l\ s \mid l\ s$):

$$\begin{aligned} &|\,1\;0\mid 0\;0\mid 0\;0\,\rangle \\ +&|\,2\;1\mid 0\;0\mid 0\;0\,\rangle \\ +&|\,2\;0\mid 1\;0\mid 0\;0\,\rangle \\ +&|\,3\;1\mid 1\;0\mid 0\;0\,\rangle \\ +&|\,2\;0\mid 2\;1\mid 0\;0\,\rangle \\ +&|\,3\;1\mid 2\;1\mid 0\;0\,\rangle \\ +&\cdots \end{aligned}$$

Obtain "clock state" by factoring $|*\rangle = \sqrt{1/2}(|20\rangle + |31\rangle)$:

$$\begin{aligned} &|\,1\;0\mid 0\;0\mid 0\;0\,\rangle \\ +&|\,2\;1\mid 0\;0\mid 0\;0\,\rangle \\ +&|\;\;*\;\;\mid 1\;0\mid 0\;0\,\rangle \\ +&|\;\;*\;\;\mid 2\;1\mid 0\;0\,\rangle \\ +&|\;\;*\;\;\mid\;\;*\;\;\mid 1\;0\,\rangle \\ +&|\;\;*\;\;\mid\;\;*\;\;\mid 2\;1\,\rangle \end{aligned}$$

Open Problems

- Complete the complexity classification of (k,l)-QSAT. The cases $(2,4), (2,3), (3,3)$ remain open.
- Hardness of $1D$-QSAT on qu-d-its with $d < 11$?
  – Due to our black-box approach, we get a huge second dimension.
- Is $1D$-$(2,d)$-QSAT still QMA_1-hard?
- $1D$-$(2,3)$-QSAT instances with fully entangled ground space?

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